Ph.D. Course
Variational problems for nonlinear Schrödinger equations on metric graphs

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Abstract

Since their first appearance in physical chemistry in 1953, *networks* (or *metric graphs*) have been proposed to model almost one-dimensional ramified structures. Despite being more than sixty years old, it is within the last two decades that the theory of evolution on networks became popular, mainly driven by the ubiquity of networks in applications, from quantum mechanics to fluid dynamics, from nonlinear optics to traffic regulation. The aim of this course is to given a wide introductory overview of recent results for nonlinear Schrödinger equations on metric graphs. In particular, we will consider variational problems for the nonlinear Schrödinger energy functional

$$\frac{1}{2}\int_{\mathcal{G}}|u'|^2\,dx-\frac{1}{p}\int_{\mathcal{G}}|u|^p\,dx$$

under the mass constraint

$$\int_{\mathcal{G}}|u|^2\,dx=\mu\,,$$

on a given graph $\mathcal{G}$. The focus will be set on the minimization problem: does the above energy admit global minimizers? What is the role of $\mathcal{G}$? What is the role of the mass μ and of the nonlinearity power p? We will see that the answers to these questions are strongly sensitive to the specific properties of the graphs.

Plan of the course

Duration and mode: 15 hours, 6 lessons of 2 hours each, 1 lesson of 3 hours.

The course is planned to be held on Zoom.

Tentative program:

Lesson 1: introduction: the nonlinear Schrödinger equation: motivation, standing waves. Variational approach to the stationary problem: fixed frequency; fixed mass. Minimization of the energy in the mass–constrained space: physical motivation (Bose–Einstein condensates); general minimization strategy in the subcritical case, the linear problem, the problem at infinity.

Lessons 2–3: graphs with half–lines. The ground state problem on the line: Gagliardo–Nirenberg inequalities. General graphs with half–lines: the comparison with the line and the theory of rearrangements. Non–existence of ground states: assumption H. Existence of ground states: examples. Hints on the critical case (time permitting).

Lessons 4–5: periodic graphs. $\mathbb{Z}$–periodicity: existence of ground states. $\mathbb{Z}^2$–periodicity: the square grid. Two–dimensional Sobolev inequality. Dimensional crossover. Hints on different two–dimensional grids and on n–dimensional grids (time permitting).

Lesson 6: infinite trees. The linear problem: Poincaré inequality and inequality with remainder. Existence and non–existence of ground states. Hints on the radial problem (time permitting).

Lesson 7: perspectives on future developments: nonlinear delta interactions for graphs with half–lines; persistence and failure of the dimensional crossover on defected grids; uniqueness of ground states.

Prerequisites

Essentially none, though basic notions of functional analysis and calculus of variations would be helpful (but not mandatory).